On the Superselection Theory of the Weyl Algebra for Diffeomorphism Invariant Quantum Gauge Theories

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Abstract

Much of the work in loop quantum gravity and quantum geometry rests on a mathematically rigorous integration theory on spaces of distributional connections. Most notably, a diffeomorphism invariant representation of the algebra of basic observables of the theory, the Ashtekar-Lewandowski representation, has been constructed. This representation is singled out by its mathematical elegance, and up to now, no other diffeomorphism invariant representation has been constructed. This raises the question whether it is unique in a precise sense.

In the present article we take steps towards answering this question. Our main result is that upon imposing relatively mild additional assumptions, the AL-representation is indeed unique. As an important tool which is also interesting in its own right, we introduce a C^*-algebra which is very similar to the Weyl algebra used in the canonical quantization of free quantum field theories.

1 Introduction

Canonical, background independent quantum field theories of connections [1] play a fundamental role in the program of canonical quantization of general relativity (including all types of matter), sometimes called loop quantum gravity or quantum general relativity. For a review geared to mathematical physicists see [2], for a general overview [3]).

The classical canonical theory can be formulated in terms of smooth connections A on principal $G-$bundles over a $D-$dimensional spatial manifold Σ for a compact gauge group G and smooth sections of an associated (under the adjoint representation) vector bundle of Lie$(G)-$valued vector densities E of weight one. The pair (A, E) coordinatizes an infinite dimensional symplectic manifold $(\mathcal{M}, \Sigma)$ whose (strong) symplectic structure s is such that A and E are canonically conjugate.

In order to quantize $(\mathcal{M}, s)$, it is necessary to smear the fields A, E. This has to be done in such a way, that the smearing interacts well with two fundamental automorphisms of the principal $G-$bundle, namely the vertical automorphisms formed by $G-$gauge transformations and the horizontal automorphisms formed by Diff(Σ) diffeomorphisms. These requirements naturally lead to holonomies and electric fluxes, that is,

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exponentiated (path-ordered) smearings of the connection over $1-$dimensional submanifolds e of Σ as well as smearings of the electric field over $(D-1)-$dimensional submanifolds S

$$h_e[A] = \mathcal{P}\exp\int_e A, \qquad E_{S,f}[E] = \int_S *E_I f^I$$

These functions on $\mathcal{M}$ generate a closed Poisson$^*-$algebra $\mathcal{P}$ and separate the points of $\mathcal{M}$. They do not depend on a choice of coordinates nor on a background metric. Therefore, diffeomorphisms and gauge transformations act on these variables in a remarkably simple way: Let φ be a diffeomorphism of Σ, then

$$\alpha_\varphi(h_e) = h_{\varphi^{-1}e}, \qquad \alpha_\varphi(E_{S,f}) = E_{\varphi^{-1}S,\varphi^* f}.$$

Similarly let $g : \Sigma \to G$ be a gauge transformation, then

$$\alpha_g(h_e) = g(a)h_e g^{-1}(b), \qquad \alpha_g(E_{S,f}) = E_{S,g^{-1}fg}$$

where a is the starting point of e and b the endpoint.

Quantization now means to promote $\mathcal{P}$ to an abstract $^*-$algebra $\mathfrak{A}$ and to look for its representations. However, for physical reasons we are not interested in arbitrary representations but those fulfilling the following criteria:

i) *Irreducibility*
The representation space $\mathcal{H}_\pi$ should contain no proper invariant subspaces, i.e. the span of vectors $\pi(a)v$ should be dense in $\mathcal{H}_p i$ for any vector $v \in \mathcal{H}$.

Irreducible representations are the building blocks of the representation theory. If their structure is clarified, more general representations can be constructed from and analyzed in terms of them.

ii) *Diffeomorphism and Gauge Invariance*
Diffeomorphism and gauge transformations are fundamental symmetries of the theory, so if we do not consider a scenario of spontaneous symmetry breaking, they should be symmetries of the ground state of the quantum theory as well.

Thus in our setting we require that there is a at least one symmetric state Ω_π in the representation space. More precisely, for the expectation value $\omega_\pi(.) :=< \Omega_\pi, .\Omega_\pi >_{\mathcal{H}_\pi}$ in that state, we require invariance:

$$\omega_\pi \circ \alpha_\varphi = \omega_\pi, \qquad \omega_\pi \circ \alpha_g = \omega_\pi$$

for all diffeomorphisms φ and gauge transformations g.

It is remarkable that so far only one representation has been found which satisfies our assumptions: This is the Ashtekar – Isham – Lewandowski representation π_0 on a Hilbert space $\mathcal{H}_0 = L_2(\overline{\mathcal{A}}, d\mu_0)$ where $\overline{\mathcal{A}}$ is the Ashtekar – Isham space of distributional connections (the spectrum of a certain Abelian C$^*-$algebra) and μ_0 is the Ashtekar – Lewandowski measure. Historically, first Ashtekar and Isham [7] were looking for a natural distributional extension $\overline{\mathcal{A}}$ of the space $\mathcal{A}$ of smooth connections, which could serve as the support for gauge invariant measures. Then Ashtekar and Lewandowski found a natural, cylindrical measure [8] which was shown to have a unique $\sigma-$additive extension μ_0 by Marolf and Mourão [9]. This measure turned out to be diffeomorphism invariant. More general diffeomorphism invariant measures were found by Baez [10], however, in contrast to μ_0 they are not faithful. That the resulting Hilbert space $\mathcal{H}_0$ indeed carries a representation of the holonomy – flux algebra was shown only later in [1], essentially that representation π_0 results by having connections and electric fields respectively act as multiplication and functional derivative operators respectively.

The present work was inspired by the question whether the fact that $\mathcal{H}_0$ is the only representation found so far which satisfies our criteria in fact means that it is the unique representation. In this article we show that upon imposing two additional and rather technical conditions on the representations, the question can be answered affirmatively: Under these assumptions, the Ashtekar-Lewandowski representation is indeed unique.

Work towards settling this questions has begun in [11], however the results obtained there rest on assumptions that exclude the interesting cases, most notably that of a noncommutative gauge group. However it might still be interesting for the reader to take a look at [11] since the discussion there is much less burdened by the technical subtleties that arise in the general case.

During the completion of this article, a very interesting work has been published by Okolow and Lewandowski [27] that aims at settling the very same question raised in this article. Their method of proof and in part also their assumptions differ from the ones used in the present article, so it is very instructive to compare the two approaches. The hope is that combining methods of the present paper with those of [27] enables one to prove a completely general and satisfactory uniqueness theorem.

Before we conclude this introduction, let us discuss the subtleties that arise due to our general setting as well as the additional assumptions we are going to make.

The first problem that arises comes from the fact that the flux operators are unbounded and so one has to worry about domain problems. In our approach, we will try to circumvent these problems by not working with the fluxes directly but with their exponentiated counterparts. More precisely, we will consider the abstract Weyl algebra formed from holonomies and exponentiated electric fluxes and represent them as bounded operators on a Hilbert space . This algebra can be equipped with a C^*-norm so that $\mathfrak{A}$ turns into a C^*-algebra and we therefore have the powerful representation theory of C^*-algebras at our disposal. However, we will require that the representations under considerations will be weakly continuous for the unitary groups generated by exponentiated fluxes. Therefore their selfadjoint generators, the fluxes themselves, will be well defined operators. In the case of an Abelian gauge group, this approach enables us to completely circumvent any specification of the domains of the fluxes. Due to technical complications for non-Abelian gauge groups, we will however have to make such a specification in that case. This is the first of the two requirements in addition to i) and ii) above that we make in order to prove our uniqueness result.

It is interesting to note that, at least for the case of an Abelian gauge group, our theorem could be compared to von Neumann's theorem [12] (uniqueness of weakly continuous, irreducible representations of the Weyl C$^*-$algebra of the phase space $(\mathcal{M} = \mathbb{R}^{2N}, \sigma = \sum_{a=1}^{N} dp_a \wedge dq^a)$ with $N < \infty$ up to unitary equivalence) since it also makes use of irreducibility and continuity. The surprise is that our theorem holds for an infinite number of degrees of freedom and that continuity is required only for one half of the variables (in fact, connections only form an affine space and not a vector space, so continuity of holonomies is even hard to formulate) while in background dependent quantum field theories we are faced with an uncountably infinite number of unitarily inequivalent representations of the canonical commutation relations [13]. There, a unique representation is usually selected by using Lorentz invariance and a specific dynamics, in that sense it is a *dynamical uniqueness*. However, while we use spatial diffeomorphism invariance, in our case we do not make use of any particular dynamics such as the Hamiltonian constraint of quantum general relativity [14] and in that sense it is a *kinematical uniqueness*.

This comparison leads us to the second subtlety and the corresponding additional assumption: The requirements i) and ii) guarantee that the the action of the automorphisms on algebra elements can be unitarily implemented in the representation. However there is a priory little control about the details of the action of these unitary operators in the Hilbert space. We will point out that there is a "natural" way for them to act in the representation Hilbert space, and we will require that this natural action is realized in the representations we consider. This is, however, a priori not the most general possibility. Two scenarios can be envisioned: In the first, one can actually show that the natural action is in fact the only possible one, and then our uniqueness result would be general. In the second, there are actually other viable unitary actions of the diffeomorphisms, and this in turn might lead to a classifications of the representations studied in terms of unitary representations of the diffeomorphism group. The picture then would be very similar to that obtained in the case of free quantum field theories, where Poincare invariant representations can be classified by unitary representations of the Poincare group. Both scenarios would be very interesting in their own ways, and we happily await a future settling of this question.

To summarize, a completely general and satisfactory picture of the diffeomorphism and gauge invariant representations of the algebra of holonomies and fluxes has not yet emerged. However, the results of the present work and that of [27, 11] point to the fact that diffeomorphism invariance is an extremely strong requirement and could mean that in background independent quantum field theories there is much less quantization freedom than in background dependent ones.

To finish, let us give an overview of the structure of the rest of present work:

In section 2 we recall from [1] the essentials of the classical formulation of canonical, background independent theories of connections, that is, the symplectic manifold $(\mathcal{M}, \sigma)$ and the corresponding classical Poisson$^*-$algebra $\mathcal{P}$ generated by holonomies and electric fluxes.
In section 3 we define the abstract $^*-$algebra $\mathfrak{A}$ and recall from [11] the general representation theory of $\mathfrak{A}$.
In section 4 we implement irreducibility and spatial diffeomorphism invariance and prove our uniqueness theorem.

2 Preliminaries

Let Σ be an analytic, connected and orientable $D-$dimensional manifold and G a compact, connected gauge group. A principal $G-$bundle P over Σ is determined by its local trivializations $\phi_I : U_I \times G \to P$ subordinate to an atlas $\{U_I\}$ of Σ. These give rise to local, smooth $G-$valued functions $g_{IJ} : U_I \cap U_J \to G$ on Σ, called transition function cocycles. A connection over P can be thought of as a collection $\{A_I\}$ of smooth, Lie$(G)-$valued one-forms over the respective charts U_I subject to the gauge covariance condition $A_I = -dg_{IJ}g_{IJ}^{-1} + \mathrm{Ad}_{g_{IJ}}(A_J)$ over $U_I \cap U_J$. The space of smooth connections $\mathcal{A}$ over P therefore depends on the bundle P but we will abuse notation in not displaying this dependence.

Similarly, we define a vector bundle E_P associated to P under the adjoint representation whose typical fiber is a Lie$(G)-$valued $(D-1)-$form on P. An electric field is a local section of E_P which we may think of as a collection $\{E_I\}$ of Lie$(G)-$valued $(D-1)-$forms on Σ subject to the gauge covariance condition $E_I = \mathrm{Ad}_{g_{IJ}}(E_J)$ over $U_I \cap U_J$. The space of smooth electric fields $\mathcal{E}$ over P depends on P as well but the dependence is also not displayed.

The space $\mathcal{A}$ can be given the structure of a manifold modeled on a Banach space in the usual way (see e.g. [15, 1]). Consider now the cotangent bundle $\mathcal{M} := T^*(\mathcal{A})$. Since $\mathcal{A}$ is a Banach manifold, also $\mathcal{M}$ is and, moreover, we may identify $\mathcal{E}$ with the sections of $\mathcal{M}$ together with the induced topology. The cotangent bundle $\mathcal{M} = \mathcal{A} \times \mathcal{E}$ can be equipped with the following (strong, see e.g. [16]) symplectic structure: Let $\mathrm{Tr} :\ \mathrm{Lie}(G) \times \mathrm{Lie}(G) \mapsto \mathbb{C}$ be a natural Ad_G-invariant metric on Lie(G) then there is a natural pairing $\mathcal{E} \times \mathcal{A} \to \mathbb{C}$ defined by

The Poisson bracket is uniquely defined by

The algebra $\mathcal{P}'$ is, however, not what we are interested in for several reasons:

i) *Gauge Invariance*
The objects $F(A), E(f)$ depend heavily on our choice of trivialization of P. It will be very hard to construct gauge invariant quantities from them, in which we are ultimately interested. In order to do that, we must work with basic functions on $\mathcal{M}$ which are different from the canonical functions $F(A), E(f)$. Of course, these problems could be avoided by fixing a gauge, however, there is no canonical gauge and most gauges are plagued by the Gribov problem.

ii) *Background Independence*
Even when ignoring the just mentioned problems, it is rather hard to construct spatially diffeomorphism invariant (background independent) representations of $\mathcal{P}'$, in fact, to the best of our knowledge such representations have not been constructed. To see where the problem is, suppose that we want to construct a representation of the form $\mathcal{H} = L_2(\mathcal{S}', d\mu)$ where $\mathcal{S}'$ is the space of tempered distributions on Σ (that is, the topological dual of the space $\mathcal{S}$ of functions of rapid decrease) and μ is a measure thereon. This is the form of the representation for free field theories [17]. Notice that the nuclear topology on $\mathcal{S}$ does not refer to any background structure except for the differentiable structure of Σ, so there is no problem up to this point. The problem arises when we define the measure μ via its generating functional $\mu(F) := \mu(\exp(iF(.)))$. For instance, if μ is a (generalized) free (Gaussian) measure, then $\mu(F) = \exp(-F(C \cdot F)/2)$ where C is a *background metric dependent* appropriate covariance which is needed in order to contract indices in the appropriate way. Interacting measures in more than three spacetime dimensions have not been constructed so far.

A solution to the first problem was suggested for canonical quantum Yang-Mills theories already by Gambini et. al. [18] and for loop quantum gravity by Jacobson, Rovelli and Smolin [19]. The idea is to work with holonomies and electric fluxes. We will explain in detail what we mean by that, because it will be important for what follows. For more details, see [2].

Definition 2.1.
i) $\mathcal{C}$ is the set of piecewise analytic, continuous, oriented, compactly supported, parameterized curves embedded in Σ. We denote by $b(c), f(c)$ the beginning and final point of c and consider the range $r(c)$ as the image of the compact interval $[0,1]$ under c.
ii) If $b(c_2) = f(c_1)$ we define composition $(c_1 \circ c_2)(t) = c_1(2t)$ if $t \in [0, \frac{1}{2}]$ and $(c_1 \circ c_2)(t) = c_2(2t-1)$ if $t \in [\frac{1}{2}, 1]$. Inversion is defined by $c^{-1}(t) := c(1-t)$.
iii) We call $c, c' \in \mathcal{C}$ equivalent, $c \sim c'$, iff c, c' differ by a finite number of reparameterizations and retracings (a segment of a curve of the form $s^{-1} \circ s'$). The set of equivalence classes p in $\mathcal{C}$ is denoted as the set of paths $\mathcal{Q}$. The functions b, f and the operations $\circ, ^{-1}$ extend from $\mathcal{C}$ to $\mathcal{Q}$.
iv) An edge $e \in \mathcal{Q}$ is a path for which an entire analytic representative $c_e \in \mathcal{C}$ exists. For edges the function r extends as $r(e) := r(c_e)$.
v) An oriented graph γ is determined by a finite number of edges $e \in E(\gamma)$ which intersect at most in their boundaries, called the vertex set $V(\gamma)$.

It is important to realize that in contrast to $\mathcal{C}$ the set $\mathcal{Q}$ is a groupoid with objects the points $x \in \Sigma$ and with the sets of morphisms given by $\mathrm{Mor}(x, y) = \{p \in \mathcal{Q};\ b(p) = x,\ f(p) = y\}$. The notion of paths is motivated by the algebraic properties of the holonomy.

Definition 2.2.
For $A \in \mathcal{A}$ and $p \in \mathcal{Q}$ we define $A(p) := h_{A,p}(1)$ where $h_{A,p} :\ [0,1] \to G$ is uniquely defined by the parallel transport equation

The worry is of course, that $A(p)$ is smeared only in one dimension rather than three such as $F(A)$ was. In order to still obtain a well-defined Poisson algebra, the electric field therefore must be smeared in at least $D-1$ dimensions. This can be done as follows:
Let S be an open, connected, simply connected, analytic, oriented, compactly supported $(D-1)-$dimensional submanifold of Σ, called a surface in what follows, let $x_0 \in S$ and for $x \in S$ let $c_{x_0,x} \in \mathcal{C}$ with $b(c_{x_0,x}) = x_0,\ f(c_{x_0,x}) = x,\ r(c_{x_0,x}) \subset S$. Then we define

For the purposes of this paper we will make also the following additional technical assumption: Notice that if $S = S_1 \cup S_2$ is the disjoint union of surfaces then we have $E_n(S) = E_n(S_1) + E_n(S_2)$. Thus we know the flux $E_n(S)$ if we know it for every connected surface S. If S is a connected surface we can triangulate it *into* $(D-1)-$*simplices* Δ *and we have* $E_n(S) = \sum_\Delta E_n(\Delta)$ *even if the different* Δ *overlap in faces, since they are of measure zero. Now each* $(D-1)-$*simplex can be decomposed into* D*,* $(D-1)-$*dimensional, cubes by choosing an interior point of* Δ*, connecting it with an interior point of each of its boundary* $(D-2)-$*simplices, connecting those points with an interior point of each of its boundary* $(D-3)-$*simplices etc. Thus, we know each* $E_n(S)$ *if we know it for each* $E_n(\Box)$ *where* $\Box$ *is a* $(D-1)-$*cube. The assumption that we now make is the following: We choose precisely one* $(D-2)-$*face of* $\Box$ *open while all others are closed. In other words, if* $\bar{\Box}$ *denotes the closure of* $\Box$ *and* $\bar{F}$ *the closure of one of its faces* F *then* $\Box = \bar{\Box} - \overline{F}$*. The classical fluxes satisfy* $E_n(\Box) = E_n(\bar{\Box})$ *so this seems to be an innocent assumption. However, it will turn out to be crucial in the quantum theory. From now on we allow only compactly supported, analytical, oriented surfaces* S *which can be written as a disjoint union* $S = \cup_\Box \Box$ *of such cubes* $\Box$ *with the specified boundary properties. Since the classical flux* $E_n(S)$ *through any* S *can be written as a limit of fluxes through those special* S*, there is no loss of generality on the classical side. The decisive feature of such a cube* $\Box$ *is that we can choose a closed* $(D-2)-$*surface* S *such that* $\Box = \Box_1 \cup \Box_2$ *is a disjoint union with* $S \subset \Box_1$*,* $S \cap \Box_2 = \emptyset$*,* $\overline{\Box_1} \cap \overline{\Box_2} = S$ *and all three* $\Box, \Box_1, \Box_2$ *are analytically diffeomorphic. The reason for why this is important will become obvious only in section 4. In order that this works, we must restrict to* $D \geq 2$ *in what follows. We feel that this assumption is not crucial for our result to hold, however, it avoids tedious case by case considerations of the intersection structure of surfaces. Thus, we ask whether the functions* $A(p), E_n(S)$ *generate a well-defined Poisson algebra which is induced from (**??**). The answer is as follows [20]:*

Definition 2.3.

i)

Given a graph γ *we define* $p_\gamma : \mathcal{A} \to G^{|E(\gamma)|}$*;* $A \mapsto \{A(e)\}_{e \in E(\gamma)}$*. A function* f *is said to be cylindrical over* γ *iff there exists a function* $f_\gamma : G^{|E(\gamma)|} \to \mathbb{C}$ *such that* $f = f_\gamma \circ p_\gamma$*. The functions cylindrical over* γ *are denoted by* Cyl_γ *and the* $^*-$*algebra of cylindrical functions is defined by* $Cyl := \cup_{\gamma \in \Gamma} Cyl_\gamma$ *where* Γ *is the set of all compactly supported, oriented, piecewise analytic graphs. Notice that* $f \in Cyl_\gamma$ *implies* $f \in Cyl_{\gamma'}$ *for any* $\gamma \subset \gamma'$ *and we identify the corresponding representatives.*

ii)

The subalgebras Cyl^n*,* $n = 0, 1, 2, .., \infty$ *of* Cyl *consist of functions of the form* $f = f_\gamma \circ p_\gamma$ *where* $f_\gamma \in C^n(G^{|E(\gamma)|})$*.*

iii)

Vector fields on $\mathcal{A}$ *are defined as maps* $Y : Cyl^n \to Cyl^{n-1}$ *which satisfy the Leibniz rule and annihilate constants. We will denote them by* Vec*.*

iv)

Given an open, compactly supported, connected, simply connected, oriented, analytic surface S *and a cylindrical function* f *we can always find a graph* γ *over which it is cylindrical and which is adapted to* S *in the following sense: Any* $e \in E(\gamma)$ *belongs to precisely one of the following subsets* $E_*(\gamma)$ *of* $E(\gamma)$ *where*
$E_{out}(\gamma) = \{e \in E(\gamma);\ e \cap S = \emptyset\}$*,*
$E_{in}(\gamma) = \{e \in E(\gamma);\ e \cap \bar{S} = e\}$*,*
$E_{up}(\gamma) = \{e \in E(\gamma);\ e \cap S = b(e),\ e \text{ points into the direction of } S\}$ *and*
$E_{down}(\gamma) = \{e \in E(\gamma);\ e \cap S = b(e),\ e \text{ points into the opposite direction of } S\}$*.*
For $e \in E(\gamma)$ *we define* $\sigma(S, e) := 0$ *if* $e \in E_{out}(\gamma) \cup E_{in}(\gamma)$ *and we define a)* $\sigma(S, e) = 1$ *if* $e \in E_{up}(\gamma)$ *b)* $\sigma(S, e) = -1$ *if* $e \in E_{down}(\gamma)$*.*
We have supplemented the regularization of the flux vector field, so far only discussed for open surfaces in the literature, to the case that $b(e)$ *is a boundary point. Our condition is compatible with the additivity of fluxes. We can now define a real-valued vector field* $Y_n(S)$ *on* Cyl *by (*$f = p_\gamma^* f_\gamma$*)*

We can now compute the commutation relations among the $W_t^n(S) = \exp(tY_n(S))$ *and the* $f \in Cyl^\infty$*.*

We have

$$W_t^n(S) f (W_t^n(S))^{-1} = \sum_{m=0^\infty} \frac{t^m}{m!} [Y_n(S), f]_{(m)}$$
$$= \sum_{m=0^\infty} \frac{t^m}{m!} (Y_n(S))^m f = W_t^n(S) \cdot f \tag{2.1}$$

where the bracket notation denotes the multiple commutator and the last line denotes the application of the exponentiated vector field to a cylindrical function. Let now $f = p_\gamma^* f_\gamma$*. Since the* R_e^j *are mutually commuting we have*

$$[W_t^n(S) \cdot f](A) = [p_\gamma^* \prod_{e \in E(\gamma)} e^{t\sigma(S,e) n_j(b(e)) R_e^j} f_\gamma](A)$$
$$= f_\gamma(\{e^{t\sigma(S,e) n^j(b(e)) \tau_j} A(e)\}_{e \in E(\gamma)}) \tag{2.2}$$

To see the equality in the last line of (2.2) it is obviously sufficient to show it for one copy of G, that is

Finally we compute the commutator of Weyl-operators by explicitly using the germ vector fields. We have

That representation is given by $\mathcal{H}_0 = L_2(\overline{\mathcal{A}}, d\mu_0)$ *where* $\overline{\mathcal{A}}$ *is the spectrum of the* C^*-*subalgebra of* $\mathfrak{A}$ *given by Cyl and* μ_0 *is a regular Borel probability measure on* $\overline{\mathcal{A}}$ *consistently defined by*

3 General Representation Theory of $\mathfrak{A}$

Let us clarify what we mean by a representation of $\mathfrak{A}$*.*

Definition 3.1. *By a representation of* $\mathfrak{A}$ *we mean an* $^*-$*algebra homomorphism* $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ *from* $\mathfrak{A}$ *into the algebra of bounded operators of a Hilbert space* $\mathcal{H}$*. Thus* $\pi(a + zb) = \pi(a) + z\pi(b),\ \pi(ab) = \pi(a)\pi(b),\ \pi(a^*) = [\pi(a)]^\dagger$ *for all* $a, b \in \mathfrak{A}, z \in \mathbb{C}$*.*

The representation theory of $\mathfrak{A}$ *is very rich and first steps towards a classification have been made in [11]. An elementary result is the following.*

Lemma 3.1.
The representation space $\mathcal{H}$ *of* $\mathfrak{A}$ *is necessarily a direct sum of Hilbert spaces*

It follows that

Notice that while the subalgebra Cyl_b *has no off-diagonal entries, that is in general not the case for the* $\pi(W_t^n(S)$*. Also, while* π_ν *is a cyclic representation for* Cyl_b*,* π *is not necessarily cyclic for* Cyl_b*, one will generically assume it to be cyclic for the full algebra* $\mathfrak{A}$ *only (that is, there is a vector* $\Omega \in \mathcal{H}$ *such that the set of states given by* $\pi(a)\Omega,\ a \in \mathfrak{A}$ *is dense in* $\mathcal{H}$*). In what follows we will only consider representations which are cyclic for* $\mathfrak{A}$ *(otherwise we can decompose* π *further into cyclic representations by the above theorem, hence cyclic representations are the basic building blocks).*

This all that one can say so far about general representations of $\mathfrak{A}$ *without making further assumptions. To get further structural control over the representation theory one must examine restricted situations of physical interest. In the next section we will study the important class of diffeomorphism invariant representations which are those realized in nature (nature is diffeomorphism invariant, so there is no need to study other representations at all, at least from a physics point of view).*

4 Diffeomorphism Invariant Representations of $\mathfrak{A}$ and a Uniqueness Theorem

The group $Diff^\omega(\Sigma)$ of analytic diffeomorphisms on Σ has a natural representation as outer automorphisms on $\mathfrak{A}$ defined for any $\varphi \in Diff^\omega(\Sigma)$ by

$$\begin{aligned} \alpha_\varphi(p^*_\gamma f_\gamma) &= p^*_{\varphi^{-1}(\gamma)} f_\gamma \\ \alpha_\varphi(W^n_t(S)) &= W^{n\circ\varphi}_t(\varphi^{-1}(S)) \end{aligned} \tag{4.1}$$

and extended by the automorphism property $\alpha_\varphi(ab) = \alpha_\varphi(a)\alpha_\varphi(b),\ \alpha_\varphi(a+zb) = \alpha_\varphi(a) + z\alpha_\varphi(b)$. It is trivial to check that $\alpha_\varphi \circ \alpha_{\varphi'} = \alpha_{\varphi\circ\varphi'}$.

Likewise, the set $Fun(\Sigma, G)$, which forms a group under pointwise multiplication, has a natural representation as outer automorphisms on $\mathfrak{A}$ defined for any $g \in Fun(\Sigma, G)$ by

$$\begin{aligned} [\alpha_g(p^*_\gamma f_\gamma)](A) &= f_\gamma(\{g(b(e))A(e)g(f(e))^{-1}) \\ \alpha_g(W^n_t(S)) &= W^{n^g}_t(S) \end{aligned} \tag{4.2}$$

where $n^g(x) = Ad_{g(x)}(n(x)),\ n(x) = n_j\tau_j$. As one can check, with these definitions we have $\alpha_\varphi \circ \alpha_g \circ \alpha_{\varphi^{-1}} = \alpha_{\varphi^* g}$ so that the combined kinematical gauge group acquires the structure of a semidirect product $\mathcal{G} = Fun(\Sigma, G) \rhd Diff^\omega(\Sigma)$ if we define $\alpha_{(g,\varphi)} := \alpha_g \circ \alpha_\varphi$ with $Fun(\Sigma, G)$ as invariant subgroup.

Definition 4.1.
i)
A cyclic representation π of $\mathfrak{A}$ is said to be diffeomorphism invariant provided that there is a unitary representation

A natural starting point for cyclic invariant representations exists, provided one manages to find a positive linear functional ω on $\mathfrak{A}$ with the invariance property

Using the language of the present paper, in [11] the following result was established.

Theorem 4.1.
Suppose that
1) $G = U(1)$.
2) π is cyclic already for Cyl_b so that necessarily $\mathcal{H} = L_2(\overline{\mathcal{A}}, d\mu)$ with cyclic vector $\Omega = 1$ by lemma 3.1.
3) π is diffeomorphism invariant with Ω as invariant cyclic vector where the diffeomorphisms act by pull back.
4) The one parameter subgroups $t \mapsto \pi(W^n_t(S))$ are weakly continuous.
5) Ω is in the domain of any self-adjoint generator $-i[\frac{d}{dt}]_{t=0}\pi(W^n_t(s))$.
Then necessarily $\mathcal{H} = L_2(\overline{\mathcal{A}}, d\mu_0) = \mathcal{H}_0$ is the Ashtekar – Lewandowski representation.

Several of the assumptions of theorem 4.1 are unsatisfactory: First of all, the restriction to $U(1)$ makes it of limited physical relevance since in particular loop quantum gravity would need such a result for general compact groups. Next, it is not natural to require that already Cyl_b is cyclic for the representation, the most general interesting representations will be those for which only the full algebra $\mathfrak{A}$ is cyclic. Furthermore, while it is natural to assume that the constants are in the domain of the self-adjoint generators of the Weyl elements (because the unit function is a cyclic vector for Cyl_b), one has no intuition whether there are not more general representations which violate this assumption.

On the other hand, if one does not assume weak continuity of the fluxes then the requirements will be too weak to limit the number of possible representations. This is already the case for the Schrödinger *representation of ordinary quantum mechanics: If one gives up weak continuity of the Weyl elements then many more representations exist which are not captured by the Stone – von Neumann theorem. In fact, the Stone – von Neumann theorem not only requires the representation to be cyclic but even to be irreducible (that is, every vector is cyclic), otherwise also more representations result. We thus expect to find a strong result also only in the irreducible case. Irreducibility is actually more physical than cyclicity since then no non-trivial invariant subspaces exist and moreover, there are no distinguished cyclic elements. We do not know at present whether cyclicity is actually enough for the result to be proved below.*

There is one more unnatural assumption in theorem 4.1: Why should it be the case that the vector 1 *is left invariant by* $U_\pi(\varphi)$ *? If we have only cyclicity of* $\mathfrak{A}$ *then it is also not clear why it should be the vector* 1 *which is cyclic. Actually, this discussion leads to the representation theory of* $Diff^\omega(\Sigma)$ *as the following discussion reveals:*
Suppose that we do have a diffeomorphism invariant representation π*. Then the action of* $U_\pi(\varphi)$ *is known on the whole representation space* $\mathcal{H}$ *provided we know it on the* 1^ν *because*

To see that there really is an abundance of unitarily inequivalent representations of $Diff^\omega(\Sigma)$*, suppose that we start from a a representation* π *of* $\mathfrak{A}$ *on a Hilbert space* $\mathcal{H}$ *with unitary pull-back representation* U_π *of* $Diff^\omega(\Sigma)$*. Let* $W \in \mathcal{B}(\mathcal{H})$ *be any bounded operator with bounded inverse which we consider as being of the form* $W = \pi(a)$ *for some* $a \in \mathfrak{A}$*. Let us also denote* $\alpha_\varphi(W) := \pi(\alpha_\varphi(a))$*. We claim that*

One might think that one can bring more structure into the analysis by requiring that the representation U_π *to be irreducible as well (not only* π*) because then it follows from Schur's lemma that* $W = \lambda id_{\mathcal{H}}$ *and unitary equivalence requires* $|\lambda| = 1$*. However, it is well known that interesting representations of the diffeomorphism group are generically quite reducible. For instance, the pull back representation on* $\mathcal{H}_0$ *is extremely reducible [1], we have a countably (under suitable superselection criteria [22]) infinite direct sum decomposition*

These cautionary remarks are just to indicate that there are a priory many inequivalent, unitary representations of $Diff^\omega(\Sigma)$ *available and their classification goes beyond the scope of the present paper. The selection of one of them might be comparable to the selection of a definite spin representation of the Poincaré group, however, it is much more complicated (the diffeomorphism group is an infinite dimensional group !). Accordingly, we must be modest and specify the representation of* $Diff^\omega(\Sigma)$ *in the statement of our theorem below. Obviously, we will choose the pull-back representation which is natural because it is available in any representation of* $\mathfrak{A}$ *as shown in lemma 3.1.*

Before we state our theorem, let us define the notion of a spin network function on $\overline{\mathcal{A}}$*.*

Definition 4.2.
Choose precisely on representative ρ *from each equivalence class of irreducible representations of* G*, denote by* d_ρ *the dimension of the representation space of* ρ *and denote for any* $h \in G$ *and* $M, N = 1,..,d_\rho$ *by* $\rho_{MN}(h)$ *the matrix elements of the unitary matrix* $\rho(h)$*. Consider a graph* γ *together with a labeling of each of its edges* $e \in E(\gamma)$ *with label* $\rho_e, M_e, N_e;\ M_e, N_e = 1,..,d_{\rho_e}$ *and collect them into a* **spin network**

Now for any $\psi \in \mathcal{H}$

Since T_s *is a continuous (on* $\overline{\mathcal{A}}$*) cylindrical function over* $\gamma(s)$ *and right translation* $A(e) \mapsto e^{tn^j\tau_j}A(e)$ *is continuous in* t*, it follows that (**??**) can be made arbitrarily small by suitably restricting the support of* ϕ*.*
$\square$

Corollary 4.1.
The set of vectors $\pi(f)1^{\nu}_{\phi,S_n}$, as $f \in Cyl^{\infty}, \nu$ vary, form a dense set of C^{∞} vectors for the self-adjoint generator $\pi(E(S_n))$ of $\pi(W_t(S_n))$, more precisely

If $n_j(x) = const.$ we may also get rid of the $n-$dependence of the ψ_{ϕ,S_n} as follows: We notice that each flux vector field can be uniquely split as $Y_n(S) = Y_n^+(S) - Y_n^-(S)$ where

We will now construct successively analytic diffeomorphisms $\varphi_k,\ k = 1,..,m$ which preserve e such that $\varphi_k^{-1}(p_l) = p_l,\ l = 0,..,k-1,\ \varphi_k^{-1}(p'_k) = p_k,\ \varphi_k^{-1}(p_{m+1}) = p_{m+1}$ where $p_0 = b(e),\ p_{m+1} = f(e)$ and such that the σ_k are not changed. Then

To construct φ_k explicitly, choose w.l.g. an analytic coordinate system such that e coincides with the interval $[0,1]$ of the x^1-axis (if e does not lie entirely within the domain of a chart, replace e by a closed segment of it that does in what follows). Then $p_k = (x_k, 0,..,0)$ and $p'_k(y_k, 0,..,0)$ are the coordinates of the points in question and we label them in such a way that $x_0 = 0 < x_1 < .. < x_m < x_{m+1} = 1,\ 0 < y_1 < .. < y_m < 1$. The situation for φ_k is such that $y_l = x_l,\ l = 1,..,k-1$ already while the $y_l,\ l = k,..,m$ are unspecified. Thus, the idea is to construct an analytic vector field $x \mapsto v_k(x)$ on $\mathbb{R}$ which has zeroes at the points $x_0 = 0, x_1,..,x_{k-1}, x_{m+1} = 1$ and whose flow maps y_k to x_k. Consider the analytic vector field on Σ defined in our coordinate system by $\vec{v}_k(\vec{x}) = (v_k(x^1), 0,..,0)$. The integral curves $c^{\vec{v}_k}_{\vec{x}}(t)$ it generates defines a one parameter family of analytic diffeomorphisms $\varphi^{\vec{v}_k}_t(\vec{x}) := c^{\vec{v}_k}_{\vec{x}}(t)$ of the form

Our ansatz is given for $\delta > 0$ by

Consider the function $f(t,x) := v_k(x)$ which does not depend explicitly on t. We have

Notice that after applying the $k-1$th diffeomorphism φ_{k-1} we have already achieved that $y_l = x_l,\ l = 1,..,k-1$ (the order of the points y_k cannot be changed by a diffeomorphism) and our job is now to construct φ_k which has to move $y_k > y_{k-1} = x_{k-1}$ to x_k while leaving $x = 0, x_1,..,x_{k-1}, 1$ fixed. Define now

From the continuity of the solution $x(t)$ we conclude that there exists $t_k \in [0,T]$ such that $x(t) = x_k$. Thus, if $t \mapsto \varphi_{k,t}$ is the one parameter family of diffeomorphisms generated by $-v_k$ we define $\varphi_k := \varphi_{k,t_k}$ and have $\varphi_k(x_l) = x_l,\ l = 0,..,k-1,\ l = m$ and $\varphi_k(y_k) = x_k$ as desired, which concludes our induction step.
$\Box$

Step 3:
*This step contains the main argument in our proof. The self-adjoint generator $\pi(Y_n(S))$ is symmetric and has dense domain $\mathcal{D}(S_n)$ for any n. From (**??**) we find the symmetry condition*

$$\begin{aligned}
& < \pi(f)1^{\nu}_{\phi,S_n}, \pi(E(S_n))\pi(f')1^{\nu'}_{\phi',S_n} > \\
= \; & i < \pi(f)1^{\nu}_{\phi,S_n}, \pi(Y(S_n)f')1^{\nu'}_{\phi',S_n} + \pi(f')1^{\nu'}_{-\dot{\phi}',S_n} > \\
= \; & < \pi(E(S_n))\pi(f)1^{\nu}_{\phi,S_n}, \pi(f')1^{\nu'}_{\phi',S_n} > \\
= \; & -i < \pi(Y(S_n)f)1^{\nu}_{\phi,S_n} + \pi(f)1^{\nu}_{-\dot{\phi},S_n}, \pi(f')1^{\nu'}_{\phi',S_n} > \qquad (4.3)
\end{aligned}$$

Choose $f=1$ then we obtain the master condition

$$
\begin{aligned}
& -<1^{\nu}_{\phi,S_n},\pi(Y(S_n)f')1^{\nu'}_{\phi',S_n}> \\
= \; & <1^{\nu}_{\phi,S_n},\pi(f')1^{\nu'}_{-\dot{\phi}',S_n}>+<1^{\nu}_{-\dot{\phi},S_n},\pi(f')1^{\nu'}_{\phi',S_n}>
\end{aligned} \tag{4.4}
$$

By similar methods

*In order to get such squares we just have to iterate (4.4) or (**??**) by choosing $f'=Y_n(S)f$ or $f'=Y^+_j(S)f$ for some f to be suitably chosen. This results in*

$$
\begin{aligned}
& <1^{\nu}_{\phi,S_n},\pi(Y(S_n)^2 f)1^{\nu'}_{\phi',S_n}> \\
= \; & <1^{\nu}_{\phi,S_n},\pi(f)1^{\nu'}_{\ddot{\phi}',S_n}>+2<1^{\nu}_{-\dot{\phi},S_n},\pi(f)1^{\nu'}_{-\dot{\phi}',S_n}>+<1^{\nu}_{\ddot{\phi},S_n},\pi(f)1^{\nu'}_{\phi',S_n}>
\end{aligned} \tag{4.5}
$$

and

$$
\begin{aligned}
& <1^{\nu}_{\phi,S},\pi((Y^+_n(S))^2 f)1^{\nu'}_{\phi',S}> \\
= \; & <1^{\nu}_{\phi,S},\pi(f)1^{\nu'}_{(R_n)^2\phi',S_n}>+2<1^{\nu}_{-R_n\phi,S},\pi(f)1^{\nu'}_{-R_n\phi',S}>+<1^{\nu}_{R_n^2\phi,S},\pi(f)1^{\nu'}_{\phi',S}>
\end{aligned} \tag{4.6}
$$

*Let us now choose $f=\alpha^{-1}_{\varphi^{e,S}_{m,\sigma}}(T_s)$ for any spin network function T_s where $\varphi^{e,S}_{m,\sigma}$ is the diffeomorphism constructed in lemma **??** such that $\varphi^{e,S}_{m,\sigma}(S)$ intersects $\gamma(s)$ precisely in m interior points p_k of $e\in E(\gamma(s))$ with relative orientation σ_k. Let us write $e=f_1^{-1}\circ e_1\circ f_2^{-1}\circ e_2\circ..\circ f_m^{-1}\circ e_m$ where $p_k=f_k\cap e_k$. Then*

$$
\begin{aligned}
& \alpha_{\varphi^{e,S}_{m,\sigma}}(Y_n(S)^2)T_s \\
= \; & \sum_{I,J=1}^{m}\sigma_I\sigma_J(R^n_{e_I}-R^n_{f_I})(R^n_{e_J}-R^n_{f_J})T_s \\
= \; & \sum_{I=1}^{m}[2(R^n_{e_I})^2+2(R^n_{f_I})^2]T_s+8\sum_{I<J}^{m}\sigma_I\sigma_J R^n_{e_I}R^n_{e_J}T_s
\end{aligned} \tag{4.7}
$$

where in the second step we have used gauge invariance of T_s at p_k, that is, $(R^n_{e_I}+R^n_{f_I})T_s=0$. Now the idea would be to choose $n^k=\delta_{jk}$, to sum over j and to average over the 2^m possible choices for $\sigma=(\sigma_1,..,\sigma_m)$. Thus

We make the general ansatz

Due to the unitarity of $\pi(W_t(S_n))$ we compute

$$
\begin{aligned}
1 \; = \; & ||\pi(W_t(S_n))1^{\nu}||^2=||M_t(S_n)1^{\nu}||^2=||\sum_{\nu'}([M_t(S_n)]_{\nu'\nu}\cdot 1)1^{\nu'}||^2 \\
= \; & \sum_{\nu'}||[M_t(S_n)]_{\nu'\nu}||^2_{\mu_{\nu'}}
\end{aligned} \tag{4.8}
$$

for any ν. Thus all the matrix entries $[M_t(S_n)]_{\nu\nu'}$ are $L_2(\overline{\mathcal{A}},d\mu_0)$ functions and we can expand them in terms of spin network functions

From diffeomorphism covariance (remember that U_π is the natural representation of the diffeomorphism group) we have for $n_j(x) = n_j = const.$

$$\begin{aligned} U_\pi(\varphi)\pi(W^n_t(S))U_\pi(\varphi)^{-1} &= \pi(\alpha_\varphi(W^n_t(S))) = \pi(W^n_t(\varphi^{-1}(S)) = M^n_t(\varphi^{-1}(S))[W^n_t(\varphi^{-1}(S)) \otimes \pi(1)] \\ &= [U_\pi(\varphi)M_t(S_n)U_\pi(\varphi)^{-1}]\,[U_\pi(\varphi)[W^n_t(S) \otimes \pi(1)]U_\pi(\varphi)^{-1}] \\ &= [U_\pi(\varphi)M_t(S_n)U_\pi(\varphi)^{-1}]\,[W^n_t(\varphi^{-1}(S)) \otimes \pi(1)] \end{aligned} \tag{4.9}$$

and

Suppose now that $\gamma(s) \neq \emptyset$ for $D > 3$ or $\gamma(s) \neq \emptyset, \partial S$ for $D = 3$ (neither the empty graph nor the graph formed by the boundary of the closure of S, that is $\gamma(s) \neq \overline{S} - Int(S)$) or $\gamma(s) \neq \emptyset, \overline{S}$ for $D = 2$. Then we find a countably infinite number of analytic diffeomorphisms φ_k which leave S invariant but such that the $\varphi_k(\gamma(s))$ are mutually different. To construct such a diffeomorphism for $D > 1$, simply take any analytical vector field which is everywhere tangent to S and tangent to ∂S (e.g. vanishes on (non differentiable points of) ∂S). Then $S, \partial S$ are left invariant as sets, but not pointwise, by the one parameter group of analytical diffeomorphisms generated by that vector field. Thus for $D > 3$ even a graph which lies completely within the closure $\overline{S}$ can be mapped non-trivially, for $D = 2$ the graph cannot be mapped non-trivially only if $\gamma(s) = S$ and for $D = 3$ we must have $\gamma(s) = \partial S$. Thus, unless one of the cases indicated holds, we always find a one parameter group of analytical diffeomorphisms $t \mapsto \varphi^t$ which preserve S but move $\gamma(s)$ non-trivially for each t and we just need to take $\varphi_k = \varphi^{1/k}$. But this implies that

We conclude that $M^n_t(S)$ is a matrix of cylindrical L_2-functions over the graph ∂S in $D = 3$ or over S in $D = 2$ and it is a constant function in $D > 3$. Hence we may write it in the form

Let us now consider the cases $D = 2, 3$ more closely. Since by construction our Weyl algebra of fluxes is built from the fluxes through a disjoint union of cubes $\Box$, the associated $\pi(W^n_t(\Box))$ are mutually commuting and it will be sufficient to consider each $\Box$ separately. We may write

Since $A \in \overline{\mathcal{A}}$ is arbitrary we find that for arbitrary $h_1, h_2 \in G$

Since G is compact, ρ^n_t, represented on that Hilbert space is unitarily equivalent to a (possibly uncountably) direct sum of irreducible, finite dimensional representations [25] (proposition 2.5 and theorem 3.1) all of which must be commutative. If G is not Abelian, then the only commutative irreducible representations are trivial and it follows immediately $\rho^n_t(h) = \pi(1)$ for all $h \in G$. If G is Abelian then $G = U(1)^N$ for some N and every irreducible representation is of the form $(u_1, .., u_N) \mapsto (u_1^{z_1}, .., u_N^{z_N})$ for some integers z_k and any $u_k \in U(1)$. In our case the representation of every $U(1)$ factor that occurs in the decomposition of $\rho^n_t(h)$ into irreducibles is therefore of the form $u \mapsto u^{z^n_t}$ where $z^n_t \in \mathbb{Z}$ and $u \in U(1)$. Due to the representation property $\pi(W^n_s(\Box))\pi(W^n_t(\Box)) = \pi(W^n_{s+t}(\Box))$ for all $s, t \in \mathbb{R}$ and due to the fact that all edges in question are of the "in" or "out" type with respect to S we infer that $\rho^n_s(h)\rho^n_t(h) = \rho^n_{s+t}(h)$ is a one-parameter group of representations. This implies that $z^n_{s+t} = z^n_s + z^n_t$ for any $s, t \in \mathbb{R}$. Due to weak continuity we have $\rho^n_t(h) \to \pi(1)$ as $t \to 0$. Since z^n_t is an integer, there exists $\epsilon^n > 0$ such that $z^n_t = 0$ for all $|t| < \epsilon^n$. But then for any $t \in \mathbb{R}$ we find $m \in \mathbb{N}$ such that $|t/m| < \epsilon^n$ and thus $z^n_t = m\, z^n_{t/m} = 0$. Thus, also in the Abelian case the only occurring representation is trivial and we also get here that $\rho^n_t(h) = \pi(1)$.

It remains to discuss the case $D > 3$. Since in this case $M_t^n(\Box) = M_t^n(\hat{\Box})$ is just a constant we have by splitting $\Box = \Box_1 \cup \Box_2$ into disjoint pieces that

We conclude that $M_t^n(S) = \pi(1)$ for $n^j(x) = n^j = const.$ and any (allowed) surface S. For an arbitrary unit vector $n_j(x)$ we find a constant unit vector n_j^0 and an element $g_{n,n^0} \in Fun(\Sigma, G)$ such that $\alpha_{g_{n,n^0}}(W_t^n(S)) = W_t^{n^0}(S)$. Then

$$\begin{aligned} \pi(W_t^n(S)) &= \pi(\alpha_{g^{-1}_{n,n^0}}(W_t^{n_0}(S))) = U_\pi(g_{n,n^0})^{-1}\pi(W_t^{n_0}(S))U_\pi(g_{n,n^0}) \\ &= U_\pi(g_{n,n^0})^{-1}[W_t^{n_0}(S) \otimes \pi(1)]U_\pi(g_{n,n^0}) = [\alpha_{g^{-1}_{n,n^0}}(W_t^{n_0}(S)) \otimes \pi(1)] = [W_t^n(S) \otimes \pi(1)] \end{aligned} \tag{4.10}$$

so that $M_t^n(S) = \pi(1)$ also in the general case.

We thus have shown that $\pi(W_t^n(S)) = W_t^n(S) \otimes \pi(1)$. We can now finally invoke irreducibility: If the representation is to be irreducible, then every vector is cyclic, in particular any of the 1^ν is cyclic. But the algebra of operators generated by $\pi(f), \pi(W_t^n(S))$ never leaves the sector $\mathcal{H}_\nu = \mathcal{H}_0 \otimes 1^\nu$. It follows that we can allow only one copy of the Ashtekar-Lewandowski Hilbert space. That $\mathcal{H}_0$ itself is the representation space of an irreducible representation of $\mathfrak{A}$ will be shown in [26].

This finishes the proof.
$\Box$

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